Two–parameter non–linear spacetime perturbations: gauge transformations and gauge invariance

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Abstract. An implicit fundamental assumption in relativistic perturbation theory is that there exists a parametric family of spacetimes that can be Taylor expanded around a background. The choice of the latter is crucial to obtain a manageable theory, so that it is sometime convenient to construct a perturbative formalism based on two (or more) parameters. The study of perturbations of rotating stars is a good example: in this case one can treat the stationary axisymmetric star using a slow rotation approximation (expansion in the angular velocity Ω), so that the background is spherical. Generic perturbations of the rotating star (say parametrized by λ) are then built on top of the axisymmetric perturbations in Ω. Clearly, any interesting physics requires non–linear perturbations, as at least terms $\lambda\Omega$ need to be considered. In this paper we analyse the gauge dependence of non–linear perturbations depending on two parameters, derive explicit higher order gauge transformation rules, and define gauge invariance. The formalism is completely general and can be used in different applications of general relativity or any other spacetime theory.

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1. Introduction

An implicit fundamental assumption in relativistic perturbation theory is that there exists a parametric family of spacetimes such that the perturbative formalism is built as a Taylor expansion of this family around a background. The perturbations are then defined as the derivative terms of this series, evaluated on this background [1]. In most cases of interest one deals with an expansion in a single parameter, which can either be a formal one, as in cosmology [2] or in the study of quasi–normal modes of stars and black holes [3, 4], or can have a specific physical meaning, as in the study of binary black hole mergers via the close limit approximation [5, 6], or in the study of quasi–normal mode excitation by a physical source (see [4, 7] and references therein).‡

In some physical applications it may be instead convenient to construct a perturbative formalism based on two (or more) parameters, because the choice of background is crucial in having a manageable theory. The study of perturbations of stationary axisymmetric rotating stars (see [10, 11, 12] and references therein) is a good example. In this case, an analytic stationary axisymmetric solution is not known, at least for reasonably interesting equations of state. A common procedure is to treat axisymmetric stars using the so–called slow rotation approximation, so that the background is a star with spherical symmetry [13, 14]. In this approach the first order in Ω discloses frame dragging effects, with the star actually remaining spherical; Ω^2 terms carry the effects of rotation on the fluid.§ This approximation is valid for angular velocities Ω much smaller than the mass shedding limit $\Omega_K \equiv \sqrt{M/R^3_{star}}$, with typical values for neutron stars $\Omega_K \sim 10^3 Hz$. Therefore the slow rotation approximation, despite the name, can still be valid for large angular velocities. In practice, the perturbative approach up to Ω^2 is accurate for most astrophysical situations, with the exception of newly born neutron stars (see [15] and references therein).

Given that the differential operators appearing in the perturbative treatment of a problem are those defined on the background, the theory is considerably simplified when the latter is spherical. Generic time dependent perturbations of the rotating star (parametrized by a dummy parameter λ and describing oscillations) are then built on top of the stationary axisymmetric perturbations in Ω. Clearly, in this approach any interesting physics requires non–linear perturbations, as at least terms of order $\lambda\Omega$ need to be considered. A similar approach could be used to study perturbations of the slowly rotating collapse, even if in specific cases [16, 17, 18] the perturbative expansion depends by one parameter only.

Studies in the literature have not analysed in full the gauge dependence and gauge invariance of the two–parameter perturbation theory. For example, in [16] the second order perturbations are treated in a gauge invariant fashion on top of the first order perturbation in a given specific gauge. The perturbation variables used are therefore non gauge invariant under a complete second order gauge transformation [2, 19], but only invariant under "first order transformations acting at second order" [16]. While this may be perfectly satisfactory from the point of view of obtaining physical results, one may wish to convert results in a given gauge to a different one [2, 20], to compare results obtained in two different gauges, or to construct a fully gauge invariant formalism. To this end one needs to know the gauge transformation rules and the rules for gauge invariance, either *up to* order n [2] or *at* order n *only*, as in [16].

‡ In the examples cited above the perturbative expansion stops at the first order, but recent interesting developments deal with second order perturbation theory [8, 9].

§ This is intuitive from a Newtonian point of view, as rotational kinetic energy goes like Ω^2.

The situation is going to be more complicated in the case of two parameters, as we shall see.

In this paper we keep in mind the above practical examples, but we do not make any specific assumption on the background spacetime and the two–parameter family it belongs to. As in [2, 21, 22], we do not even need to assume that the background is a solution of Einstein's field equations: the formalism is completely general and can be applied to any spacetime theory. We analyse the gauge dependence of perturbations in the case when they depend on two parameters, λ and Ω, derive explicit gauge transformation rules up to fourth order, i.e. including any term $\lambda^k\Omega^{k'}$ with $k+k'\leq 4$, and define gauge invariance. This choice of keeping fixed the total perturbative order is due to the generality of our approach. In practical applications one would be guided by the physical characteristics of the problem in deciding where to truncate the perturbative expansion. For example, in the case of a rotating star one could consider first order oscillations, parametrized by λ, on top of a stationary axisymmetric background described up to Ω^2, neglecting therefore $\lambda^2\Omega$ terms. Or instead, one could decide that $\lambda^2\Omega$ terms are more interesting than the $\lambda\Omega^2$ ones in certain cases.

From a practical point of view, our aim is to derive the effects of gauge transformations on tensor fields T up to order $k+k'=4$. It is indeed reasonable to assume that in a practical example like that of rotating stars, at most one will want to consider second order oscillations $\sim\lambda^2$ on top of a slowly rotating background described up to $O(\Omega^2)$, in order to take into account large oscillations and fluid deformations due to rotation.

We will show that the coordinate form of a two–parameter gauge transformation is represented by:

$$\begin{aligned}\tilde{x}^\mu &= x^\mu + \lambda\xi^\mu_{(1,0)} + \Omega\xi^\mu_{(0,1)}\\ &\quad + \frac{\lambda^2}{2}\left(\xi^\mu_{(2,0)} + \xi^\nu_{(1,0)}\xi^\mu_{(1,0),\nu}\right) + \frac{\Omega^2}{2}\left(\xi^\mu_{(0,2)} + \xi^\nu_{(0,1)}\xi^\mu_{(0,1),\nu}\right)\\ &\quad + \lambda\Omega\left(\xi^\mu_{(1,1)} + \epsilon_0\xi^\nu_{(1,0)}\xi^\mu_{(0,1),\nu} + \epsilon_1\xi^\nu_{(0,1)}\xi^\mu_{(1,0),\nu}\right) + O^3(\lambda,\Omega)\,, \end{aligned} \tag{1}$$

where the full expression is given in Eq. (72). Here $\xi^\mu_{(1,0)}$, $\xi^\mu_{(0,1)}$, $\xi^\mu_{(2,0)}$, $\xi^\mu_{(1,1)}$, and $\xi^\mu_{(0,2)}$ are independent vector fields and (ϵ_0,ϵ_1) are any two real numbers satisfying $\epsilon_0+\epsilon_1=1$. Coupling terms like the $\lambda\Omega$ in (1) are the expected new features of the two–parameter case, cf. [2, 21]. Our main results are the explicit transformation rules for the perturbations of a tensor field T and the conditions for the gauge invariance of these perturbations.

The paper is organized as follows: in Section 2 we develop the necessary mathematical tools, deriving Taylor expansion formulae for two–parameter groups of diffeomorphisms and for general two–parameter families of diffeomorphisms. In Section 3 we set up an appropriate geometrical description of the gauge dependence of perturbations in the specific case of two–parameter families of spacetimes. In Section 4 we apply the tools developed in Section 2 to the framework introduced in Section 3, in order to define gauge invariance and formulas for gauge transformations, up to fourth order in the two–parameter perturbative expansion. Section 5 is devoted to the conclusions. We follow the notation used previously in [2, 21, 22] for the case of one parameter perturbations.

2. Taylor expansion of tensor fields

In order to consider the issues of gauge transformations and gauge invariance in two–parameter perturbation theory we need first to introduce some mathematical tools concerning the two–parameter Taylor expansion of tensor fields. Since Taylor expansions are aimed to provide the value of a quantity at some point in terms of its value, and the value of its derivatives, at another point, a Taylor expansion of tensorial quantities can only be defined through a mapping between tensors at different points of the spacetime manifold $\mathcal{M}$. In this section we consider the cases where such a mapping is given by a two–parameter family of diffeomorphisms of $\mathcal{M}$, starting from the simplest case in which such a family constitutes a group.

2.1. Two–parameter groups of diffeomorphisms

Given a differentiable manifold $\mathcal{M}$, a two–parameter group of diffeomorphisms ϕ of $\mathcal{M}$ can be represented as follows

$$\begin{array}{rccc} \phi: & \mathcal{M}\times\mathrm{I\!R}^2 & \longrightarrow & \mathcal{M} \\ & (p,\lambda,\Omega) & \longmapsto & q=\phi_{\lambda,\Omega}(p)\,, \end{array} \tag{2}$$

and has the following property

$$\phi_{\lambda_1,\Omega_1}\circ\phi_{\lambda_2,\Omega_2}=\phi_{\lambda_1+\lambda_2,\Omega_1+\Omega_2}\,,\quad \forall\lambda,\Omega\in\mathrm{I\!R}\,. \tag{3}$$

For the purposes of this paper, it will be useful to decompose it in two one–parameter groups of diffeomorphisms (flows) that remain implicitly defined by the equalities

$$\phi_{\lambda,\Omega}=\phi_{\lambda,0}\circ\phi_{0,\Omega}=\phi_{0,\Omega}\circ\phi_{\lambda,0}\,, \tag{4}$$

that follow from (3). The flows $\phi_{\lambda,0}$ and $\phi_{0,\Omega}$ are generated by two vector fields, η and ζ respectively, acting on the tangent space of $\mathcal{M}\times\mathrm{I\!R}^2$. The Lie derivatives of a generic tensor T with respect to η and ζ are

$$\mathcal{L}_\eta T:=\lim_{\lambda\to0}\frac{1}{\lambda}(\phi^*_{\lambda,0}T-T)=\left[\frac{d}{d\lambda}\phi^*_{\lambda,0}T\right]_{\lambda=0}=\left[\frac{\partial}{\partial\lambda}\phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0}\,, \tag{5}$$

$$\mathcal{L}_\zeta T:=\lim_{\Omega\to0}\frac{1}{\Omega}(\phi^*_{0,\Omega}T-T)=\left[\frac{d}{d\Omega}\phi^*_{0,\Omega}T\right]_{\Omega=0}=\left[\frac{\partial}{\partial\Omega}\phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0}\,, \tag{6}$$

where the superscript * denotes the pull–back map associated with the corresponding diffeomorphism [2]. Because of the last equality‖ in (4) the group is Abelian:

$$[\eta,\zeta]=0\,. \tag{7}$$

The Taylor expansion of the pull–backs $\phi^*_{\lambda,0}T,\phi^*_{0,\Omega}T$ is given by (see [2])

$$\phi^*_{\lambda,0}T=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\left[\frac{d^k}{d\lambda^k}\phi^*_{\lambda,0}T\right]_{\lambda=0}=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}^k_\eta T\,, \tag{8}$$

$$\phi^*_{0,\Omega}T=\sum_{k=0}^{\infty}\frac{\Omega^k}{k!}\left[\frac{d^k}{d\Omega^k}\phi^*_{0,\Omega}T\right]_{\Omega=0}=\sum_{k=0}^{\infty}\frac{\Omega^k}{k!}\mathcal{L}^k_\zeta T\,. \tag{9}$$

‖ We adopt the convention that the first label in ϕ corresponds to the flow generated by η and parametrized by λ, while the second label corresponds to the flow generated by ζ and parametrized by Ω. Therefore $\phi_{\lambda,\Omega}\neq\phi_{\Omega,\lambda}$.

From this, using (4), we can derive the Taylor expansion of the two–parameter group of pull–backs $\phi^*_{\lambda,\Omega}T$:

$$\phi^*_{\lambda,\Omega}T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k\Omega^{k'}}{k!\,k'!}\left[\frac{\partial^{k+k'}}{\partial\lambda^k\partial\Omega^{k'}}\phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} = \sum_{k,k'=0}^{\infty}\frac{\lambda^k\Omega^{k'}}{k!\,k'!}\mathcal{L}^k_\eta\mathcal{L}^{k'}_\zeta T\,. \tag{10}$$

2.2. Two–parameter family of diffeomorphisms

Let us now consider the general case of a two–parameter family of diffeomorphisms Φ:

$$\begin{array}{rccc} \Phi: & \mathcal{M}\times\mathbb{R}^2 & \longrightarrow & \mathcal{M} \\ & (p,\lambda,\Omega) & \longmapsto & q=\Phi_{\lambda,\Omega}(p)\,. \end{array} \tag{11}$$

In this general case

$$\Phi_{\lambda_1,\Omega_1}\circ\Phi_{\lambda_2,\Omega_2}\neq\Phi_{\lambda_1+\lambda_2,\Omega_1+\Omega_2}\,, \tag{12}$$

which means that the diffeomorphisms $\Phi_{\lambda,\Omega}$ do not form in general a group, and hence we cannot decompose $\Phi_{\lambda,\Omega}$ as in the case of a two–parameter group of diffeomorphisms. The Taylor expansion of the pull–back of $\Phi_{\lambda,\Omega}$ is formally given by

$$\Phi^*_{\lambda,\Omega}T = \sum_{k,k'=0}^{\infty}\frac{\lambda^k\Omega^{k'}}{k!\,k'!}\left[\frac{\partial^{k+k'}}{\partial\lambda^k\partial\Omega^{k'}}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0}\,. \tag{13}$$

Since the diffeomorphisms $\Phi_{\lambda,\Omega}$ do not form a group we cannot write this expansion directly in terms of Lie derivative operators as in the previous case, Eq. (10). Nevertheless, in view of the gauge transformations to be derived in Section 4, we would like to find an alternative way of expressing the expansion (13) in terms of suitable Lie derivatives, in a similar way as it was done in [2, 22] in the one–parameter case. To this end, we are going to introduce a set of operators (the set needed to produce a two–parameter perturbative expansion up to order $\lambda^k\Omega^{k'}$, with $k+k'=4$) by giving their actions on a general tensorial quantity T:

$$\mathcal{L}_{(1,0)}T := \left[\frac{\partial}{\partial\lambda}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0}\,, \tag{14}$$

$$\mathcal{L}_{(0,1)}T := \left[\frac{\partial}{\partial\Omega}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0}\,, \tag{15}$$

$$\mathcal{L}_{(2,0)}T := \left[\frac{\partial^2}{\partial\lambda^2}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} - \mathcal{L}^2_{(1,0)}T\,, \tag{16}$$

$$\mathcal{L}_{(1,1)}T := \left[\frac{\partial^2}{\partial\lambda\partial\Omega}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} - \left(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}+\epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\right)T\,, \tag{17}$$

$$\mathcal{L}_{(0,2)}T := \left[\frac{\partial^2}{\partial\Omega^2}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} - \mathcal{L}^2_{(0,1)}T\,, \tag{18}$$

$$\mathcal{L}_{(3,0)}T := \left[\frac{\partial^3}{\partial\lambda^3}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} - 3\mathcal{L}_{(1,0)}\mathcal{L}_{(2,0)}T - \mathcal{L}^3_{(1,0)}T\,, \tag{19}$$

$$\begin{aligned}\mathcal{L}_{(2,1)}T := {} & \left[\frac{\partial^3}{\partial\lambda^2\partial\Omega}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\ & - 2\mathcal{L}_{(1,0)}\mathcal{L}_{(1,1)}T - \mathcal{L}_{(0,1)}\mathcal{L}_{(2,0)}T - 2\epsilon_2\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}T \\ & - (\epsilon_1-\epsilon_2)\mathcal{L}_{(0,1)}\mathcal{L}^2_{(1,0)}T - (\epsilon_0-\epsilon_2)\mathcal{L}^2_{(1,0)}\mathcal{L}_{(0,1)}T\,,\end{aligned} \tag{20}$$

$$\begin{aligned}
\mathcal{L}_{(1,2)}T := & \left[\frac{\partial^3}{\partial\lambda\partial\Omega^2}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\
& - 2\mathcal{L}_{(0,1)}\mathcal{L}_{(1,1)}T - \mathcal{L}_{(1,0)}\mathcal{L}_{(0,2)}T - 2\epsilon_3\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T \\
& - (\epsilon_0 - \epsilon_3)\mathcal{L}_{(1,0)}\mathcal{L}^2_{(0,1)}T - (\epsilon_1 - \epsilon_3)\mathcal{L}^2_{(0,1)}\mathcal{L}_{(1,0)}T\,, \qquad (21)
\end{aligned}$$

$$\mathcal{L}_{(0,3)}T := \left[\frac{\partial^3}{\partial\Omega^3}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} - 3\mathcal{L}_{(0,1)}\mathcal{L}_{(0,2)}T - \mathcal{L}^3_{(0,1)}T\,, \qquad (22)$$

$$\begin{aligned}
\mathcal{L}_{(4,0)}T := & \left[\frac{\partial^4}{\partial\lambda^4}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\
& - 4\mathcal{L}_{(1,0)}\mathcal{L}_{(3,0)}T - 3\mathcal{L}^2_{(2,0)}T - 6\mathcal{L}^2_{(1,0)}\mathcal{L}_{(2,0)}T - \mathcal{L}^4_{(1,0)}T\,, \qquad (23)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{(3,1)}T := & \left[\frac{\partial^4}{\partial\lambda^3\partial\Omega}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\
& - 3\mathcal{L}_{(1,0)}\mathcal{L}_{(2,1)}T - \mathcal{L}_{(0,1)}\mathcal{L}_{(3,0)}T - 3\epsilon_4\mathcal{L}_{(2,0)}\mathcal{L}_{(1,1)}T - 3\epsilon_5\mathcal{L}_{(1,1)}\mathcal{L}_{(2,0)}T \\
& - 3\mathcal{L}^2_{(1,0)}\mathcal{L}_{(1,1)}T - 3\left(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T + \epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\right)\mathcal{L}_{(2,0)}T \\
& - (\epsilon_1 - \epsilon_2 - \epsilon_6)\mathcal{L}_{(0,1)}\mathcal{L}^3_{(1,0)}T - 3\epsilon_6\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}^2_{(1,0)}T \\
& - 3(\epsilon_2 - \epsilon_6)\mathcal{L}^2_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}T - (\epsilon_0 - 2\epsilon_2 + \epsilon_6)\mathcal{L}^3_{(1,0)}\mathcal{L}_{(0,1)}T\,, \qquad (24)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{(2,2)}T := & \left[\frac{\partial^4}{\partial\lambda^2\partial\Omega^2}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\
& - 2\mathcal{L}_{(1,0)}\mathcal{L}_{(1,2)}T + 2\mathcal{L}_{(0,1)}\mathcal{L}_{(2,1)}T - 2\mathcal{L}^2_{(1,1)}T \\
& - \epsilon_7\mathcal{L}_{(2,0)}\mathcal{L}_{(0,2)}T - \epsilon_8\mathcal{L}_{(0,2)}\mathcal{L}_{(2,0)}T - \mathcal{L}^2_{(1,0)}\mathcal{L}_{(0,2)}T - \mathcal{L}^2_{(0,1)}\mathcal{L}_{(2,0)}T \\
& - 4\left(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T + \epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\right)\mathcal{L}_{(1,1)}T \\
& + (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9)\mathcal{L}^2_{(0,1)}\mathcal{L}^2_{(1,0)}T + (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9)\mathcal{L}^2_{(1,0)}\mathcal{L}^2_{(0,1)}T \\
& - 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 - \epsilon_9)\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T \\
& - 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 + \epsilon_9)\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}T \\
& + 2(\epsilon_3 - \epsilon_0\epsilon_1)\mathcal{L}_{(1,0)}\mathcal{L}^2_{(0,1)}\mathcal{L}_{(1,0)}T + 2(\epsilon_2 - \epsilon_0\epsilon_1)\mathcal{L}_{(0,1)}\mathcal{L}^2_{(1,0)}\mathcal{L}_{(0,1)}T\,, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{(1,3)}T := & \left[\frac{\partial^4}{\partial\lambda\partial\Omega^3}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\
& - 3\mathcal{L}_{(0,1)}\mathcal{L}_{(1,2)}T - \mathcal{L}_{(1,0)}\mathcal{L}_{(0,3)}T - 3\epsilon_{10}\mathcal{L}_{(0,2)}\mathcal{L}_{(1,1)}T - 3\epsilon_{11}\mathcal{L}_{(1,1)}\mathcal{L}_{(0,2)}T \\
& - 3\mathcal{L}^2_{(0,1)}\mathcal{L}_{(1,1)}T - 3\left(\epsilon_0\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T + \epsilon_1\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\right)\mathcal{L}_{(0,2)}T \\
& - (\epsilon_0 - \epsilon_3 - \epsilon_{12})\mathcal{L}_{(1,0)}\mathcal{L}^3_{(0,1)}T - 3\epsilon_{12}\mathcal{L}_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}^2_{(0,1)}T \\
& - 3(\epsilon_3 - \epsilon_{12})\mathcal{L}^2_{(0,1)}\mathcal{L}_{(1,0)}\mathcal{L}_{(0,1)}T - (\epsilon_1 - 2\epsilon_3 + \epsilon_{12})\mathcal{L}^3_{(0,1)}\mathcal{L}_{(1,0)}T\,, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{(0,4)}T := & \left[\frac{\partial^4}{\partial\Omega^4}\Phi^*_{\lambda,\Omega}T\right]_{\lambda=\Omega=0} \\
& - 4\mathcal{L}_{(0,1)}\mathcal{L}_{(0,3)}T - 3\mathcal{L}^2_{(0,2)}T - 6\mathcal{L}^2_{(0,1)}\mathcal{L}_{(0,2)}T - \mathcal{L}^4_{(0,1)}T\,, \qquad (27)
\end{aligned}$$

where $\epsilon_0, \ldots, \epsilon_{12}$ are real number which must satisfy the following constraints

$$\epsilon_0 + \epsilon_1 = 1\,, \quad \epsilon_4 + \epsilon_5 = 1\,, \quad \epsilon_7 + \epsilon_8 = 1\,, \quad \epsilon_{10} + \epsilon_{11} = 1\,. \qquad (28)$$

One can easily check that these operators are linear and satisfy the Leibnitz rule, and therefore they are derivatives. Then, for each of them one can define a vector field

$\xi_{(p,q)}$, such that

$$\mathcal{L}_{\xi_{(p,q)}} T := \mathcal{L}_{(p,q)} T \qquad (p, q \in \mathbb{N}) \,, \tag{29}$$

where a proof of this statement is given in Appendix A.

In the particular case when Φ is a group of diffeomorphisms we recover the previous case (subsection 2.1), and $\mathcal{L}_{(p,q)} = 0$ if $p+q > 1$.

Using the differential operators we have just introduced we can express the Taylor expansion (13) of $\Phi_{\lambda,\Omega}$, up to fourth order in λ and Ω, in terms of the Lie derivatives associated with the vector fields $\xi_{(p,q)}$ (29):

$$
\begin{aligned}
\Phi^*_{\lambda,\Omega} T = {} & T + \lambda \mathcal{L}_{\xi_{(1,0)}} T + \Omega \mathcal{L}_{\xi_{(0,1)}} T \\
& + \frac{\lambda^2}{2} \left\{ \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}} \right\} T + \frac{\Omega^2}{2} \left\{ \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}} \right\} T \\
& + \lambda\Omega \left\{ \mathcal{L}_{\xi_{(1,1)}} + \epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right\} T \\
& + \frac{\lambda^3}{6} \left\{ \mathcal{L}_{\xi_{(3,0)}} + 3\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^3_{\xi_{(1,0)}} \right\} T \\
& + \frac{\lambda^2\Omega}{2} \left\{ \mathcal{L}_{\xi_{(2,1)}} + 2\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(2,0)}} + 2\epsilon_2 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right. \\
& \left. + (\epsilon_1 - \epsilon_2) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}^2_{\xi_{(1,0)}} + (\epsilon_0 - \epsilon_2) \mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \right\} T \\
& + \frac{\lambda\Omega^2}{2} \left\{ \mathcal{L}_{\xi_{(1,2)}} + 2\mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,2)}} + 2\epsilon_3 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \right. \\
& \left. + (\epsilon_0 - \epsilon_3) \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}^2_{\xi_{(0,1)}} + (\epsilon_1 - \epsilon_3) \mathcal{L}^2_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right\} T \\
& + \frac{\Omega^3}{6} \left\{ \mathcal{L}_{\xi_{(0,3)}} + 3\mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^3_{\xi_{(0,1)}} \right\} T \\
& + \frac{\lambda^4}{24} \left\{ \mathcal{L}_{\xi_{(4,0)}} + 4\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(3,0)}} + 3\mathcal{L}^2_{\xi_{(2,0)}} + 6\mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^4_{\xi_{(1,0)}} \right\} T \\
& + \frac{\lambda^3\Omega}{6} \left\{ \mathcal{L}_{\xi_{(3,1)}} + 3\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,1)}} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(3,0)}} + 3\epsilon_4 \mathcal{L}_{\xi_{(2,0)}} \mathcal{L}_{\xi_{(1,1)}} \right. \\
& + 3\epsilon_5 \mathcal{L}_{\xi_{(1,1)}} \mathcal{L}_{\xi_{(2,0)}} + 3\mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(1,1)}} + 3\left(\epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right) \mathcal{L}_{\xi_{(2,0)}} \\
& + (\epsilon_1 - \epsilon_2 - \epsilon_6) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}^3_{\xi_{(1,0)}} + 3\epsilon_6 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}^2_{\xi_{(1,0)}} \\
& \left. + 3(\epsilon_2 - \epsilon_6) \mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} + (\epsilon_0 - 2\epsilon_2 + \epsilon_6) \mathcal{L}^3_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \right\} T \\
& + \frac{\lambda^2\Omega^2}{4} \left\{ \mathcal{L}_{\xi_{(2,2)}} + 2\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(1,2)}} + 2\mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(2,1)}} + 2\mathcal{L}^2_{\xi_{(1,1)}} \right. \\
& + \epsilon_7 \mathcal{L}_{\xi_{(2,0)}} \mathcal{L}_{\xi_{(0,2)}} + \epsilon_8 \mathcal{L}_{\xi_{(0,2)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(2,0)}} \\
& + 4\left(\epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right) \mathcal{L}_{\xi_{(1,1)}} \\
& - (\epsilon_3 + \epsilon_2 - \epsilon_1 + \epsilon_9) \mathcal{L}^2_{\xi_{(0,1)}} \mathcal{L}^2_{\xi_{(1,0)}} - (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9) \mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}^2_{\xi_{(0,1)}} \\
& + 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 - \epsilon_9) \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \\
& + 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 + \epsilon_9) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \\
& \left. - 2(\epsilon_3 - \epsilon_0\epsilon_1) \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}^2_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} - 2(\epsilon_2 - \epsilon_0\epsilon_1) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \right\} T \\
& + \frac{\lambda\Omega^3}{6} \left\{ \mathcal{L}_{\xi_{(1,3)}} + 3\mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,2)}} + \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,3)}} + 3\epsilon_{10} \mathcal{L}_{\xi_{(0,2)}} \mathcal{L}_{\xi_{(1,1)}} \right. \\
& + 3\epsilon_{11} \mathcal{L}_{\xi_{(1,1)}} \mathcal{L}_{\xi_{(0,2)}} + 3\mathcal{L}^2_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,1)}} + 3\left(\epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \right) \mathcal{L}_{\xi_{(0,2)}}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\epsilon_0 - \epsilon_3 - \epsilon_{12})\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^3_{\xi_{(0,1)}} + 3\epsilon_{12}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}} \\
&+3(\epsilon_3 - \epsilon_{12})\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + (\epsilon_1 - 2\epsilon_3 + \epsilon_{12})\mathcal{L}^3_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\Big\} T \\
&+ \frac{\Omega^4}{24}\Big\{\mathcal{L}_{\xi_{(0,4)}} + 4\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,3)}} + 3\mathcal{L}^2_{\xi_{(0,2)}} + 6\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^4_{\xi_{(0,1)}}\Big\} T \\
&+ O^5(\lambda, \Omega)\,. \qquad (30)
\end{aligned}
$$

The parameters $\{\epsilon_i\}$ are not fixed numbers, the only conditions that they must satisfy are the constraints (28). They represent the arbitrariness that we have in the *reconstruction* of the Taylor expansion of a two–parameter family of diffeomorphisms in terms of Lie derivative operators.

3. Gauges in perturbation theory and the two–parameter case

Let us consider for the moment a spacetime $\{g^{\rm b}, \mathcal{M}_0\}$ which we call the background, and a physical spacetime $\{g, \mathcal{M}\}$ which we attempt to describe as a perturbation of $\{g^{\rm b}, \mathcal{M}_0\}$.¶ In relativistic perturbation theory we are used to write expressions of the form

$$
g_{\mu\nu}(x) = g^{\rm b}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \qquad (31)
$$

relating a perturbed tensor field such as the metric with the background value of the same field and with the perturbation. In doing this, we are implicitly assigning a correspondence between points of the perturbed and the background spacetimes [23]. Indeed through (31), which is a relation between the images of the fields in $\mathbb{R}^m$ rather than between the fields themselves on the respective manifolds $\mathcal{M}$ and $\mathcal{M}_0$, we are saying that there is a unique point x in $\mathbb{R}^m$ that is at the same time the image of *two* points: one (say q) in $\mathcal{M}_0$ and one (o) in $\mathcal{M}$. This correspondence is what is usually called a gauge choice in the context of perturbation theory. Clearly, this is more than the usual assignment of coordinate labels to points of a single spacetime [24]. Furthermore, the correspondence established by relations such as (31) is not *per se* unique, but rather (31) typically defines a set of gauges, unless certain specific restrictions are satisfied by the fields involved (e.g., some metric components vanish). Leaving this problem aside, i.e. supposing that the gauge has been somehow completely fixed, let us look more precisely at the implications of (31), adopting the geometrical description illustrated in Figure 1.

If we call $\boldsymbol{X}$ the chart on $\mathcal{M}_0$ and $\boldsymbol{Z}$ the chart on $\mathcal{M}$ we see that if we choose $z(o) = x(q)$, i.e. the correspondence between points of $\{g^{\rm b}, \mathcal{M}_0\}$ and $\{g, \mathcal{M}\}$ implicit in (31), we are implicitly defining a map φ between $\mathcal{M}_0$ and $\mathcal{M}$, such that $\varphi = \boldsymbol{Z}^{-1} \circ \boldsymbol{X}$. Thus from the geometrical point of view a gauge choice is an identification of points of $\mathcal{M}_0$ and $\mathcal{M}$. Therefore, we could as well start directly assigning the point identification map φ first, calling φ itself a gauge, and defining coordinates adapted to it later. This turns out to be a simpler way of proceeding in order to derive the gauge transformations in the following section.

Let us follow this idea in the specific case of two parameters, introducing an $(m+2)$–dimensional manifold $\mathcal{N}$, foliated by m–dimensional submanifolds diffeomorphic to $\mathcal{M}$, so that $\mathcal{N} = \mathcal{M} \times \mathbb{R}^2$. We shall label each copy of $\mathcal{M}$ by the corresponding value of the parameters λ, Ω. The manifold $\mathcal{N}$ has a natural differentiable structure which

¶ As manifolds $\mathcal{M}_0$ and $\mathcal{M}$ are the same; for generality we assume that they are m–dimensional.

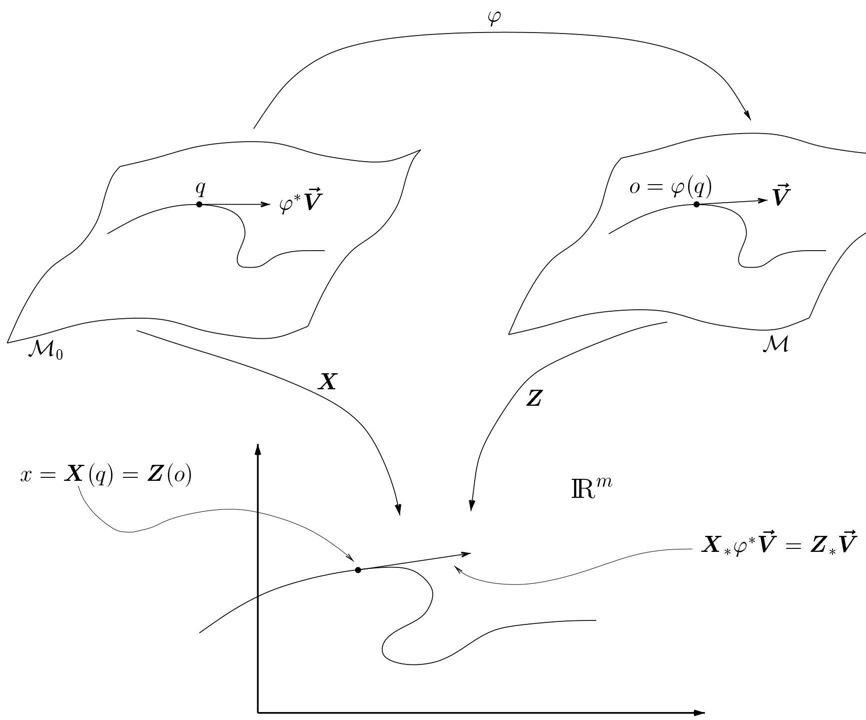

Figure 1. By choosing the coordinates on $\mathcal{M}_0$ and $\mathcal{M}$ in such a way that $(\boldsymbol{Z} \circ \varphi)^\mu = x^\mu$, a curve in $\mathcal{M}_0$, and its φ–transformed in $\mathcal{M}$ have the same representation in $\mathrm{I\!R}^m$. Therefore, the components of the tangent vectors V and $\varphi^* V$ at the points $\varphi(p)$ and p are the same: $(\varphi^* V)^\mu(x) = (\varphi^* V) x^\mu \mid_q = V(\boldsymbol{X} \circ \varphi^{-1})^\mu \mid_{\varphi(q)} = V\, z^\mu \mid_{\varphi(q)} = V^\mu(x)$.

is the direct product of those of $\mathcal{M}$ and $\mathrm{I\!R}^2$. We can then choose charts on $\mathcal{N}$ in which x^μ $(\mu = 0, 1, \ldots m-1)$ are coordinates of each leaf $\mathcal{M}_{\lambda,\Omega}$ and $x^m = \lambda$, $x^{m+1} = \Omega$.

Now, if a tensor $T_{\lambda,\Omega}$ is given on each $\mathcal{M}_{\lambda,\Omega}$, we have that a tensor field T is automatically defined on $\mathcal{N}$ by the relation $T(p, \lambda, \Omega) := T_{\lambda,\Omega}(p)$, with $p \in \mathcal{M}_{\lambda,\Omega}$.[+] In particular, on each $\mathcal{M}_{\lambda,\Omega}$ one has a metric $g_{\lambda,\Omega}$ and a set of matter fields $\tau_{\lambda,\Omega}$, satisfying the set of field equations

$$\mathcal{E}[g_{\lambda,\Omega}, \tau_{\lambda,\Omega}] = 0 \,. \tag{32}$$

Correspondingly, the fields g and τ are defined on $\mathcal{N}$.

We now want to define the perturbation in any tensor T, therefore we must find a way to compare $T_{\lambda,\Omega}$ with T_0: this requires a prescription for identifying points of $\mathcal{M}_{\lambda,\Omega}$ with those of $\mathcal{M}_0$. This is easily accomplished by assigning a diffeomorphism $\varphi_{\lambda,\Omega} : \mathcal{N} \to \mathcal{N}$ such that $\varphi_{\lambda,\Omega}\big|_{\mathcal{M}_0} : \mathcal{M}_0 \to \mathcal{M}_{\lambda,\Omega}$. Clearly, $\varphi_{\lambda,\Omega}$ can be regarded as the member of a two–parameter group of diffeomorphisms φ on $\mathcal{N}$, corresponding to the values of λ, Ω of the group parameter. Therefore, we could equally well give

[+] Tensor fields on $\mathcal{N}$ constructed in this way are “tangent” to $\mathcal{M}$, in the sense that their components m and $m+1$ in the charts we have defined vanish identically.

the vector fields ${}^{\varphi}\eta$, ${}^{\varphi}\zeta$ that generate φ. In the chart introduced above, ${}^{\varphi}\eta^m = 1$, ${}^{\varphi}\eta^{m+1} = 0$, ${}^{\varphi}\zeta^m = 0$, ${}^{\varphi}\zeta^{m+1} = 1$ but, except for these conditions, ${}^{\varphi}\eta$, ${}^{\varphi}\zeta$ remain arbitrary. For convenience, we shall also refer to such a pair of vector fields as a gauge. It is always possible to take the chart above defined such that ${}^{\varphi}\eta^\mu = {}^{\varphi}\zeta^\mu = 0$. So, in this chart, point of different folii $\mathcal{M}_{\lambda,\Omega}$ connected by the diffeomorphism φ have the same $\mathcal{M}$–coordinates $x^0, \ldots, x^{m-1}$, and differ only by the value of the coordinates λ, Ω. We call such a chart "adapted to the gauge φ": this is what is always used in practice.

The perturbation in T can now be defined simply as

$$\Delta_0 T_{\lambda,\Omega} := \varphi^*_{\lambda,\Omega} T\big|_{\mathcal{M}_0} - T_0 \,. \tag{33}$$

The first term on the right–hand side of (33) can be Taylor–expanded using (10) to get

$$\Delta_0 T_{\lambda,\Omega} = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k!\, k'!} \delta^{(k,k')} T - T_0 \,, \tag{34}$$

where

$$\delta^{(k,k')} T := \left[\frac{\partial^{k+k'}}{\partial \lambda^k \partial \Omega^{k'}} \varphi^*_{\lambda,\Omega} T \right]_{\lambda=0,\Omega=0,\mathcal{M}_0} = \mathcal{L}^k_{{}^{\varphi}\eta} \mathcal{L}^{k'}_{{}^{\varphi}\zeta} T \Big|_{\mathcal{M}_0} \,, \tag{35}$$

which defines the perturbation of order (k, k') of T (notice that $\delta^{(0,0)} T = T_0$). It is worth noticing $\Delta_0 T_{\lambda,\Omega}$ and $\delta^{(k,k')} T$ are defined on $\mathcal{M}_0$; this formalizes the statement one commonly finds in the literature that "perturbations are fields living in the background". It is important to appreciate that the parameters λ, Ω labelling the various spacetime models also serve to perform the expansion (34), and therefore determine what one means by "perturbations of order (k, k')".

4. Gauge invariance and gauge transformations

Let us now suppose that two gauges $X := ({}^{\varphi}\eta, {}^{\varphi}\zeta)$, $Y := ({}^{\psi}\eta, {}^{\psi}\zeta)$ are defined on $\mathcal{N}$, such that in the chart discussed above*

$$\begin{array}{lllll} {}^{\varphi}\eta^m & = & {}^{\psi}\eta^m & = & 1\,, \\ {}^{\varphi}\zeta^m & = & {}^{\psi}\zeta^m & = & 0\,, \\ {}^{\varphi}\eta^{m+1} & = & {}^{\psi}\eta^{m+1} & = & 0\,, \\ {}^{\varphi}\zeta^{m+1} & = & {}^{\psi}\zeta^{m+1} & = & 1\,. \end{array} \tag{36}$$

Correspondingly, their integral curves define two two–parameter groups of diffeomorphisms φ and ψ on $\mathcal{N}$, that connect any two leaves of the foliation. Thus X and Y are everywhere transverse to $\mathcal{M}_{\lambda,\Omega}$ and points lying on the same integral surface of either of the two are to be regarded *as the same point* within the respective gauge: φ and ψ are both point identification maps, i.e. two different gauge choices.

The fields X and Y can both be used to pull back a generic tensor T and therefore to construct two other tensor fields $\varphi^*_{\lambda,\Omega} T$ and $\psi^*_{\lambda,\Omega} T$, for any given value of (λ, Ω). In particular, on $\mathcal{M}_0$ we now have three tensor fields, i.e. T_0 and

$$T^X_{\lambda,\Omega} := \varphi^*_{\lambda,\Omega} T\big|_{\mathcal{M}_0} \,, \qquad T^Y_{\lambda,\Omega} := \psi^*_{\lambda,\Omega} T\big|_{\mathcal{M}_0} \,. \tag{37}$$

* In general, if the chart is adapted to the gauge φ, i.e. ${}^{\varphi}\eta^\mu = {}^{\varphi}\zeta^\mu = 0$, it is not adapted to the gauge ψ, so ${}^{\psi}\eta^\mu \neq 0$, ${}^{\psi}\zeta^\mu \neq 0$.

Since X and Y represent gauge choices for mapping a perturbed manifold $\mathcal{M}_{\lambda,\Omega}$ into the unperturbed one $\mathcal{M}_0$, $T^X_{\lambda,\Omega}$ and $T^Y_{\lambda,\Omega}$ are the representations, in $\mathcal{M}_0$, of the perturbed tensor according to the two gauges. We can write, using (33)–(35) and the expansion (10),

$$T^X_{\lambda,\Omega} = \sum_{k=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k!\,k'!} \delta_X^{(k,k')} T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k!\,k'!} \mathcal{L}^k_{{}^{\varphi}\eta} \mathcal{L}^{k'}_{{}^{\varphi}\zeta} T = T_0 + \Delta_0^{\varphi} T_{\lambda,\Omega}\,, \tag{38}$$

$$T^Y_{\lambda,\Omega} = \sum_{k=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k!\,k'!} \delta_Y^{(k,k')} T = \sum_{k,k'=0}^{\infty} \frac{\lambda^k \Omega^{k'}}{k!\,k'!} \mathcal{L}^k_{{}^{\psi}\eta} \mathcal{L}^{k'}_{{}^{\psi}\zeta} T = T_0 + \Delta_0^{\psi} T_{\lambda,\Omega} \tag{39}$$

where $\delta^{(k,k')} T^X$, $\delta^{(k,k')} T^Y$ are the perturbations (35) in the gauges X and Y respectively, i.e.

$$\begin{aligned} \delta^{(k,k')} T^X &= \left. \mathcal{L}^k_{{}^{\varphi}\eta} \mathcal{L}^{k'}_{{}^{\varphi}\zeta} T \right|_{\mathcal{M}_0}\,, \\ \delta^{(k,k')} T^Y &= \left. \mathcal{L}^k_{{}^{\psi}\eta} \mathcal{L}^{k'}_{{}^{\psi}\zeta} T \right|_{\mathcal{M}_0}\,. \end{aligned} \tag{40}$$

4.1. Gauge invariance

If $T^X_{\lambda,\Omega} = T^Y_{\lambda,\Omega}$, for any pair of gauges X and Y, we say that T is *totally gauge invariant*. This is a very strong condition, because then (38) and (39) imply that $\delta^{(k,k')} T^X = \delta^{(k,k')} T^Y$, for all gauges X and Y and for any (k, k'). In any practical case, however, one is interested in perturbations up to a fixed order. It is thus convenient to weaken the definition above, saying that T is *gauge invariant up to order* (n, n') iff for any two gauges X and Y

$$\delta^{(k,k')} T^X = \delta^{(k,k')} T^Y \qquad \forall (k, k') \quad \text{with } k \le n\,,\ k' \le n'\,. \tag{41}$$

We have that a tensor field T is gauge invariant to order (n, n') iff in a given gauge $\mathcal{L}_\xi \delta^{(k,k')} T = 0$, for any vector field ξ defined on $\mathcal{M}$ and for any $(k, k') < (n, n')$.

To prove this statement, let us first show that it is true for $(n, n') = (1, 0)$. In fact, if $\delta_X^{(1,0)} T = \delta_Y^{(1,0)} T$ for two arbitrary gauges X, Y, we have $\mathcal{L}_{{}^{\varphi}\eta - {}^{\psi}\eta} T|_{\mathcal{M}_0} = 0$. But since X and Y are arbitrary gauges, it follows that ${}^{\varphi}\eta - {}^{\psi}\eta$ is an arbitrary field ξ, and $\xi^m = \xi^{m+1} = 0$ because ${}^{\varphi}\eta^m = {}^{\psi}\eta^m = 1$, ${}^{\varphi}\eta^{m+1} = {}^{\psi}\eta^{m+1} = 0$, so ξ is tangent to $\mathcal{M}$. In the same way one proves the statement for $(n, n') = (0, 1)$. Now let us suppose that the statement is true for some (n, n'). Then, if one also has $\delta_X^{(n+1,n')} T|_{\mathcal{M}_0} = \delta_Y^{(n+1,n')} T|_{\mathcal{M}_0}$, it follows that $\mathcal{L}_{{}^{\varphi}\eta - {}^{\psi}\eta} \delta_X^{(n,n')} T = 0$, while if $\delta_X^{(n,n'+1)} T|_{\mathcal{M}_0} = \delta_Y^{(n,n'+1)} T|_{\mathcal{M}_0}$, it follows that $\mathcal{L}_{{}^{\varphi}\zeta - {}^{\psi}\zeta} \delta_X^{(n,n')} T = 0$, and we establish the result by induction over (n, n').

As a consequence, T is gauge invariant to order (n, n') iff T_0 and all its perturbations of order lower than (n, n') are, in any gauge, either vanishing or constant scalars, or a combination of Kronecker deltas with constant coefficients. Thus, this generalizes to an arbitrary order (n, n') and to the two–parameter case the results of [2, 24, 25]. Further, it then follows that T is totally gauge invariant iff it is a combination of Kronecker deltas with coefficients depending only on λ, Ω.

4.2. Gauge transformations

If a tensor T is not gauge invariant, it is important to know how its representation on $\mathcal{M}_0$ changes under a gauge transformation. To this purpose it is natural to introduce,

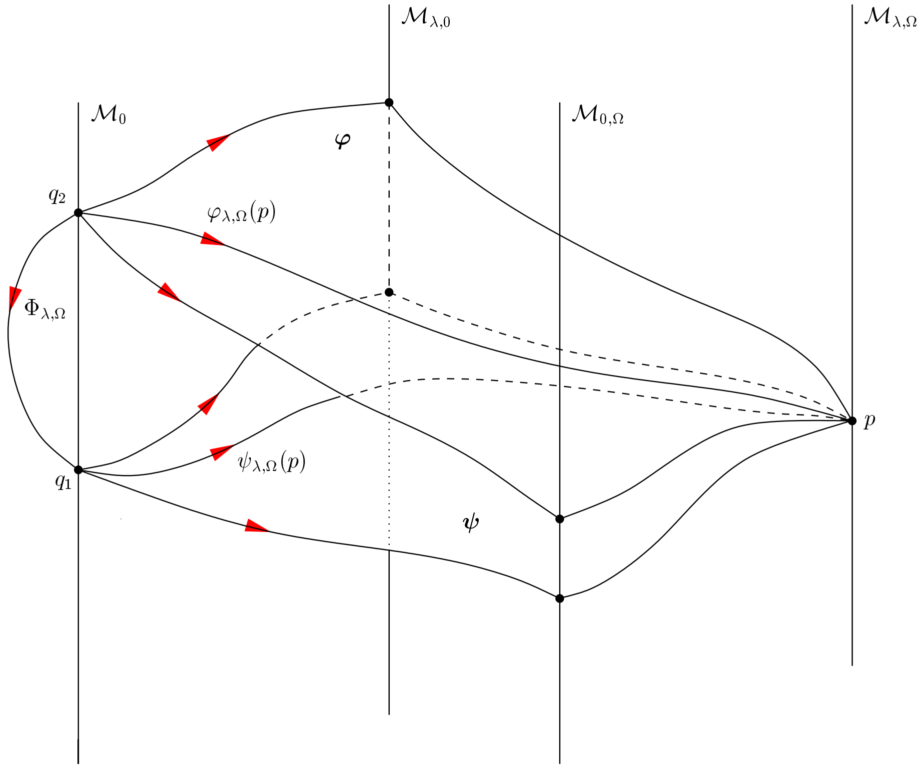

Figure 2. The action of a gauge transformation $\Phi_{\lambda,\Omega}$, represented on the background spacetime $\mathcal{M}_0$. The two gauges φ, ψ are two dimensional submanifolds embedded in $\mathcal{N} = \mathcal{M} \times \mathrm{I\!R}^2$, respectively, the lower and upper surfaces in the figure.

for each value of $(\lambda, \Omega) \in \mathrm{I\!R}^2$, the diffeomorphism $\Phi_{\lambda,\Omega} : \mathcal{M}_0 \to \mathcal{M}_0$ defined by

$$\Phi_{\lambda,\Omega} := \varphi_{-\lambda,-\Omega} \circ \psi_{\lambda,\Omega} \, . \tag{42}$$

Given that from the geometrical point of view adopted here φ and ψ are two gauges, Φ represents the gauge transformation. The action of $\Phi_{\lambda,\Omega}$ is illustrated in Fig. 2. We must stress that $\Phi : \mathcal{M}_0 \times \mathrm{I\!R}^2 \to \mathcal{M}_0$ thus defined, *is not* a two–parameter group of diffeomorphisms in $\mathcal{M}_0$. In fact, $\Phi_{\lambda_1,\Omega_1} \circ \Phi_{\lambda_2,\Omega_2} \neq \Phi_{\lambda_1+\lambda_2,\Omega_1+\Omega_2}$, essentially because the fields X and Y have, in general, a non–vanishing commutator. However, it can be Taylor expanded, using the results of section (2.2).

The tensor fields $T^X_{\lambda,\Omega}$ and $T^Y_{\lambda,\Omega}$, defined on $\mathcal{M}_0$ by the gauges φ and ψ, are connected by the linear map $\Phi^*_{\lambda,\Omega}$:

$$\begin{aligned} T^Y_{\lambda,\Omega} = {} & \psi^*_{\lambda,\Omega} \big|_{\mathcal{M}_0} = \left(\psi^*_{\lambda,\Omega} \varphi^*_{-\lambda,-\Omega} \varphi^*_{\lambda,\Omega} T \right)\big|_{\mathcal{M}_0} \\ = {} & \Phi^*_{\lambda,\Omega} \left(\varphi^*_{\lambda,\Omega} T \right)\big|_{\mathcal{M}_0} = \Phi^*_{\lambda,\Omega} T^X_{\lambda,\Omega} \, . \end{aligned} \tag{43}$$

Thus, the gauge transformation to an arbitrary order (n, n') is given by the Taylor expansion of the pull–back $\Phi^*_{\lambda,\Omega} T$, whose terms are explicitly given in section 2.2. Up to fourth order, we have explicitly from (30)

$$T^Y_{\lambda,\Omega} = T^X_{\lambda,\Omega} + \lambda \mathcal{L}_{\xi_{(1,0)}} T^X_{\lambda,\Omega} + \Omega \mathcal{L}_{\xi_{(0,1)}} T^X_{\lambda,\Omega}$$

$$
\begin{aligned}
&+\frac{\lambda^2}{2}\left\{\mathcal{L}_{\xi_{(2,0)}}+\mathcal{L}^2_{\xi_{(1,0)}}\right\}T^X_{\lambda,\Omega}+\frac{\Omega^2}{2}\left\{\mathcal{L}_{\xi_{(0,2)}}+\mathcal{L}^2_{\xi_{(0,1)}}\right\}T^X_{\lambda,\Omega}\\
&+\lambda\Omega\left\{\mathcal{L}_{\xi_{(1,1)}}+\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}+\epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda^3}{6}\left\{\mathcal{L}_{\xi_{(3,0)}}+3\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(2,0)}}+\mathcal{L}^3_{\xi_{(1,0)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda^2\Omega}{2}\left\{\mathcal{L}_{\xi_{(2,1)}}+2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,1)}}+\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,0)}}+2\epsilon_2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right.\\
&\left.+(\epsilon_1-\epsilon_2)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}}+(\epsilon_0-\epsilon_2)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda\Omega^2}{2}\left\{\mathcal{L}_{\xi_{(1,2)}}+2\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,1)}}+\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,2)}}+2\epsilon_3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right.\\
&\left.+(\epsilon_0-\epsilon_3)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}+(\epsilon_1-\epsilon_3)\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\Omega^3}{6}\left\{\mathcal{L}_{\xi_{(0,3)}}+3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,2)}}+\mathcal{L}^3_{\xi_{(0,1)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda^4}{24}\left\{\mathcal{L}_{\xi_{(4,0)}}+4\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(3,0)}}+3\mathcal{L}^2_{\xi_{(2,0)}}+6\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(2,0)}}+\mathcal{L}^4_{\xi_{(1,0)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda^3\Omega}{6}\left\{\mathcal{L}_{\xi_{(3,1)}}+3\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(2,1)}}+\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(3,0)}}+3\epsilon_4\mathcal{L}_{\xi_{(2,0)}}\mathcal{L}_{\xi_{(1,1)}}\right.\\
&+3\epsilon_5\mathcal{L}_{\xi_{(1,1)}}\mathcal{L}_{\xi_{(2,0)}}+3\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,1)}}+3\left(\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}+\epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right)\mathcal{L}_{\xi_{(2,0)}}\\
&+(\epsilon_1-\epsilon_2-\epsilon_6)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^3_{\xi_{(1,0)}}+3\epsilon_6\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}}\\
&\left.+3(\epsilon_2-\epsilon_6)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}+(\epsilon_0-2\epsilon_2+\epsilon_6)\mathcal{L}^3_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda^2\Omega^2}{4}\left\{\mathcal{L}_{\xi_{(2,2)}}+2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,2)}}+2\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,1)}}+2\mathcal{L}^2_{\xi_{(1,1)}}\right.\\
&+\epsilon_7\mathcal{L}_{\xi_{(2,0)}}\mathcal{L}_{\xi_{(0,2)}}+\epsilon_8\mathcal{L}_{\xi_{(0,2)}}\mathcal{L}_{\xi_{(2,0)}}+\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,2)}}+\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,0)}}\\
&+4\left(\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}+\epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right)\mathcal{L}_{\xi_{(1,1)}}\\
&-(\epsilon_3+\epsilon_2-\epsilon_1+\epsilon_9)\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}}-(\epsilon_3+\epsilon_2-\epsilon_0-\epsilon_9)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}\\
&+2(\epsilon_3+\epsilon_2-\epsilon_0\epsilon_1-\epsilon_9)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\\
&+2(\epsilon_3+\epsilon_2-\epsilon_0\epsilon_1+\epsilon_9)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\\
&\left.-2(\epsilon_3-\epsilon_0\epsilon_1)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}-2(\epsilon_2-\epsilon_0\epsilon_1)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\lambda\Omega^3}{6}\left\{\mathcal{L}_{\xi_{(1,3)}}+3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,2)}}+\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,3)}}+3\epsilon_{10}\mathcal{L}_{\xi_{(0,2)}}\mathcal{L}_{\xi_{(1,1)}}\right.\\
&+3\epsilon_{11}\mathcal{L}_{\xi_{(1,1)}}\mathcal{L}_{\xi_{(0,2)}}+3\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,1)}}+3\left(\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}+\epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right)\mathcal{L}_{\xi_{(0,2)}}\\
&+(\epsilon_0-\epsilon_3-\epsilon_{12})\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^3_{\xi_{(0,1)}}+3\epsilon_{12}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}\\
&\left.+3(\epsilon_3-\epsilon_{12})\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}+(\epsilon_1-2\epsilon_3+\epsilon_{12})\mathcal{L}^3_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}T^X_{\lambda,\Omega}\\
&+\frac{\Omega^4}{24}\left\{\mathcal{L}_{\xi_{(0,4)}}+4\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,3)}}+3\mathcal{L}^2_{\xi_{(0,2)}}+6\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,2)}}+\mathcal{L}^4_{\xi_{(0,1)}}\right\}T^X_{\lambda,\Omega}\\
&+O^5(\lambda,\Omega)\,,
\end{aligned}
\tag{44}
$$

where the $\xi_{(p,q)}$ are now the generators of the gauge transformation $\Phi_{\lambda,\Omega}$.

We can now relate the perturbations in the two gauges. To order (n, n') with $n + n' \leq 4$, these relations can be derived by substituting (38), (39) in (44):

$$\delta_Y^{(1,0)}T - \delta_X^{(1,0)}T = \mathcal{L}_{\xi_{(1,0)}} T_0 \,, \tag{45}$$

$$\delta_Y^{(0,1)}T - \delta_X^{(0,1)}T = \mathcal{L}_{\xi_{(0,1)}} T_0 \,, \tag{46}$$

$$\delta_Y^{(2,0)}T - \delta_X^{(2,0)}T = 2\mathcal{L}_{\xi_{(1,0)}} \delta_X^{(1,0)}T + \left\{\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\right\} T_0 \,, \tag{47}$$

$$\begin{aligned}\delta_Y^{(1,1)}T - \delta_X^{(1,1)}T &= \mathcal{L}_{\xi_{(1,0)}} \delta_X^{(0,1)}T + \mathcal{L}_{\xi_{(0,1)}} \delta_X^{(1,0)}T \\ &+ \left\{\mathcal{L}_{\xi_{(1,1)}} + \epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}\right\} T_0 \,,\end{aligned} \tag{48}$$

$$\delta_Y^{(0,2)}T - \delta_X^{(0,2)}T = 2\mathcal{L}_{\xi_{(0,1)}} \delta_X^{(0,1)}T + \left\{\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\right\} T_0 \,, \tag{49}$$

$$\begin{aligned}\delta_Y^{(3,0)}T - \delta_X^{(3,0)}T &= 3\mathcal{L}_{\xi_{(1,0)}} \delta_X^{(2,0)}T + 3\left\{\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\right\} \delta_X^{(1,0)}T \\ &+ \left\{\mathcal{L}_{\xi_{(3,0)}} + 3\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^3_{\xi_{(1,0)}}\right\} T_0 \,,\end{aligned} \tag{50}$$

$$\begin{aligned}\delta_Y^{(2,1)}T - \delta_X^{(2,1)}T &= 2\mathcal{L}_{\xi_{(1,0)}} \delta_X^{(1,1)}T + \mathcal{L}_{\xi_{(0,1)}} \delta_X^{(2,0)}T + \left\{\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\right\} \delta_X^{(0,1)}T \\ &+ 2\left\{\mathcal{L}_{\xi_{(1,1)}} + \epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}\right\} \delta_X^{(1,0)}T \\ &+ \left\{\mathcal{L}_{\xi_{(2,1)}} + 2\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(2,0)}} + 2\epsilon_2 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}\right. \\ &\left. + (\epsilon_1 - \epsilon_2) \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}^2_{\xi_{(1,0)}} + (\epsilon_0 - \epsilon_2) \mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}}\right\} T_0 \,,\end{aligned} \tag{51}$$

$$\begin{aligned}\delta_Y^{(1,2)}T - \delta_X^{(1,2)}T &= 2\mathcal{L}_{\xi_{(0,1)}} \delta_X^{(1,1)}T + \mathcal{L}_{\xi_{(1,0)}} \delta_X^{(0,2)}T + \left\{\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\right\} \delta_X^{(1,0)}T \\ &+ 2\left\{\mathcal{L}_{\xi_{(1,1)}} + \epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}\right\} \delta_X^{(0,1)}T \\ &+ \left\{\mathcal{L}_{\xi_{(1,2)}} + 2\mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,2)}} + 2\epsilon_3 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}}\right. \\ &\left. + (\epsilon_0 - \epsilon_3) \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}^2_{\xi_{(0,1)}} + (\epsilon_1 - \epsilon_3) \mathcal{L}^2_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}\right\} T_0 \,,\end{aligned} \tag{52}$$

$$\begin{aligned}\delta_Y^{(0,3)}T - \delta_X^{(0,3)}T &= 3\mathcal{L}_{\xi_{(0,1)}} \delta_X^{(0,2)}T + 3\left\{\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\right\} \delta_X^{(0,1)}T \\ &+ \left\{\mathcal{L}_{\xi_{(0,3)}} + 3\mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^3_{\xi_{(0,1)}}\right\} T_0 \,,\end{aligned} \tag{53}$$

$$\begin{aligned}\delta_Y^{(4,0)}T - \delta_X^{(4,0)}T &= 4\mathcal{L}_{\xi_{(1,0)}} \delta_X^{(3,0)}T + 6\left\{\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\right\} \delta_X^{(2,0)}T \\ &+ 4\left\{\mathcal{L}_{\xi_{(3,0)}} + 3\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^3_{\xi_{(1,0)}}\right\} \delta_X^{(1,0)}T \\ &+ \left\{\mathcal{L}_{\xi_{(4,0)}} + 4\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(3,0)}} + 3\mathcal{L}^2_{\xi_{(2,0)}} + 6\mathcal{L}^2_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^4_{\xi_{(1,0)}}\right\} T_0 \,,\end{aligned} \tag{54}$$

$$\begin{aligned}\delta_Y^{(3,1)}T - \delta_X^{(3,1)}T &= 3\mathcal{L}_{\xi_{(1,0)}} \delta_X^{(2,1)}T + \mathcal{L}_{\xi_{(0,1)}} \delta_X^{(3,0)}T + 3\left\{\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\right\} \delta_X^{(1,1)}T \\ &+ 3\left\{\mathcal{L}_{\xi_{(1,1)}} + \epsilon_0 \mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(0,1)}} + \epsilon_1 \mathcal{L}_{\xi_{(0,1)}} \mathcal{L}_{\xi_{(1,0)}}\right\} \delta_X^{(2,0)}T \\ &+ \left\{\mathcal{L}_{\xi_{(3,0)}} + 3\mathcal{L}_{\xi_{(1,0)}} \mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^3_{\xi_{(1,0)}}\right\} \delta_X^{(0,1)}T\end{aligned}$$

$$+ 3\left\{\mathcal{L}_{\xi_{(2,1)}} + 2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,0)}} + 2\epsilon_2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right.$$
$$\left.+(\epsilon_1-\epsilon_2)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}} + (\epsilon_0-\epsilon_2)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}\delta^{(1,0)}_X T$$
$$+\left\{\mathcal{L}_{\xi_{(3,1)}} + 3\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(2,1)}} + \mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(3,0)}} + 3\epsilon_4\mathcal{L}_{\xi_{(2,0)}}\mathcal{L}_{\xi_{(1,1)}} + 3\epsilon_5\mathcal{L}_{\xi_{(1,1)}}\mathcal{L}_{\xi_{(2,0)}}\right.$$
$$+ 3\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,1)}} + 3\left(\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + \epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right)\mathcal{L}_{\xi_{(2,0)}}$$
$$+ (\epsilon_1-\epsilon_2-\epsilon_6)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^3_{\xi_{(1,0)}} + 3\epsilon_6\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}}$$
$$\left.+3(\epsilon_2-\epsilon_6)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}} + (\epsilon_0-2\epsilon_2+\epsilon_6)\mathcal{L}^3_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}T_0\,, \tag{55}$$

$$\delta^{(2,2)}_Y T - \delta^{(2,2)}_X T = 2\mathcal{L}_{\xi_{(1,0)}}\delta^{(1,2)}_X T + 2\mathcal{L}_{\xi_{(0,1)}}\delta^{(2,1)}_X T$$
$$+\left\{\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\right\}\delta^{(0,2)}_X T + \left\{\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\right\}\delta^{(2,0)}_X T$$
$$+ 4\left\{\mathcal{L}_{\xi_{(1,1)}} + \epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + \epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}\delta^{(1,1)}_X T$$
$$+ 2\left\{\mathcal{L}_{\xi_{(2,1)}} + 2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,0)}} + 2\epsilon_2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right.$$
$$\left.+(\epsilon_1-\epsilon_2)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}} + (\epsilon_0-\epsilon_2)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}\delta^{(0,1)}_X T$$
$$+ 2\left\{\mathcal{L}_{\xi_{(1,2)}} + 2\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,2)}} + 2\epsilon_3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right.$$
$$\left.+(\epsilon_0-\epsilon_3)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}} + (\epsilon_1-\epsilon_3)\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}\delta^{(1,0)}_X T$$
$$+\left\{\mathcal{L}_{\xi_{(2,2)}} + 2\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(1,2)}} + 2\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,1)}} + 2\mathcal{L}^2_{\xi_{(1,1)}}\right.$$
$$+ \epsilon_7\mathcal{L}_{\xi_{(2,0)}}\mathcal{L}_{\xi_{(0,2)}} + \epsilon_8\mathcal{L}_{\xi_{(0,2)}}\mathcal{L}_{\xi_{(2,0)}} + \mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(2,0)}}$$
$$+ 4\left(\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + \epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right)\mathcal{L}_{\xi_{(1,1)}}$$
$$-(\epsilon_3+\epsilon_2-\epsilon_1+\epsilon_9)\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}} - (\epsilon_3+\epsilon_2-\epsilon_0-\epsilon_9)\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}$$
$$+ 2(\epsilon_3+\epsilon_2-\epsilon_0\epsilon_1-\epsilon_9)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}$$
$$+ 2(\epsilon_3+\epsilon_2-\epsilon_0\epsilon_1+\epsilon_9)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}$$
$$\left.-2(\epsilon_3-\epsilon_0\epsilon_1)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}} - 2(\epsilon_2-\epsilon_0\epsilon_1)\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}^2_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right\}T_0\,, \tag{56}$$

$$\delta^{(1,3)}_Y T - \delta^{(1,3)}_X T = 3\mathcal{L}_{\xi_{(0,1)}}\delta^{(1,2)}_X T + \mathcal{L}_{\xi_{(1,0)}}\delta^{(0,3)}_X T + 3\left\{\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\right\}\delta^{(1,1)}_X T$$
$$+ 3\left\{\mathcal{L}_{\xi_{(1,1)}} + \epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + \epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}\delta^{(0,2)}_X T$$
$$+\left\{\mathcal{L}_{\xi_{(0,3)}} + 3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^3_{\xi_{(0,1)}}\right\}\delta^{(1,0)}_X T$$
$$+ 3\left\{\mathcal{L}_{\xi_{(1,2)}} + 2\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,1)}} + \mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,2)}} + 2\epsilon_3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}}\right.$$
$$\left.+(\epsilon_0-\epsilon_3)\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}} + (\epsilon_1-\epsilon_3)\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}\delta^{(0,1)}_X T$$
$$+\left\{\mathcal{L}_{\xi_{(1,3)}} + 3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,2)}} + \mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,3)}} + 3\epsilon_{10}\mathcal{L}_{\xi_{(0,2)}}\mathcal{L}_{\xi_{(1,1)}} + 3\epsilon_{11}\mathcal{L}_{\xi_{(1,1)}}\mathcal{L}_{\xi_{(0,2)}}\right.$$
$$+ 3\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,1)}} + 3\left(\epsilon_0\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + \epsilon_1\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right)\mathcal{L}_{\xi_{(0,2)}}$$
$$+ (\epsilon_0-\epsilon_3-\epsilon_{12})\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^3_{\xi_{(0,1)}} + 3\epsilon_{12}\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}^2_{\xi_{(0,1)}}$$
$$\left.+3(\epsilon_3-\epsilon_{12})\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\mathcal{L}_{\xi_{(0,1)}} + (\epsilon_1-2\epsilon_3+\epsilon_{12})\mathcal{L}^3_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(1,0)}}\right\}T_0\,, \tag{57}$$

$$\delta^{(0,4)}_Y T - \delta^{(0,4)}_X T = 4\mathcal{L}_{\xi_{(0,1)}}\delta^{(0,3)}_X T + 6\left\{\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^2_{\xi_{(0,1)}}\right\}\delta^{(0,2)}_X T$$

$$+ 4\left\{\mathcal{L}_{\xi_{(0,3)}} + 3\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^3_{\xi_{(0,1)}}\right\}\delta^{(0,1)}_X T$$
$$+ \left\{\mathcal{L}_{\xi_{(0,4)}} + 4\mathcal{L}_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,3)}} + 3\mathcal{L}^2_{\xi_{(0,2)}} + 6\mathcal{L}^2_{\xi_{(0,1)}}\mathcal{L}_{\xi_{(0,2)}} + \mathcal{L}^4_{\xi_{(0,1)}}\right\} T_0 \,. \tag{58}$$

This result is, of course, consistent with the characterization of gauge invariance given in subsection 4.1. Equations (45) and (46) imply that $T_{\lambda,\Omega}$ is gauge invariant to the order $(1,0)$ or $(0,1)$ iff $\mathcal{L}_\xi T_0 = 0$, for any vector field on $\mathcal{M}_0$. Equation (47) imply that $T_{\lambda,\Omega}$ is gauge invariant to the order $(2,0)$ iff $\mathcal{L}_\xi T_0 = 0$ *and* $\mathcal{L}_\xi \delta^{(1,0)} T = 0$, for any vector field on $\mathcal{M}_0$, and so on for all the orders.

It is also possible to find the explicit expressions for the generators $\xi_{(p,q)}$ of the gauge transformation Φ in terms of the gauge vector fields $X = ({}^\varphi\eta, {}^\varphi\zeta)$ and $Y = ({}^\psi\eta, {}^\psi\zeta)$. We write here their expressions up to second order:

$$\xi_{(1,0)} = {}^\psi\eta - {}^\varphi\eta \,, \tag{59}$$
$$\xi_{(0,1)} = {}^\psi\zeta - {}^\varphi\zeta \,, \tag{60}$$
$$\xi_{(2,0)} = [{}^\varphi\eta, {}^\psi\eta] \,, \tag{61}$$
$$\xi_{(1,1)} = \epsilon_0[{}^\varphi\eta, {}^\psi\zeta] + \epsilon_1[{}^\varphi\zeta, {}^\psi\eta] \,, \tag{62}$$
$$\xi_{(0,2)} = [{}^\varphi\zeta, {}^\psi\zeta] \,. \tag{63}$$

4.3. Coordinate transformations

Up to now, we have built a two–parameter formalism using a geometrical, coordinate–free language. However, in order to carry out explicit calculations in a practical case, one has to introduce systems of local coordinates. In this respect, all our expressions are immediately translated into components simply by using the expression of the components of the Lie derivative of a tensor. Nonetheless, much of the literature on the subject is written using coordinate systems, and gauge transformations are most often represented by the corresponding coordinate transformations. For this reason, we devote this subsection to describe how to establish the translation between the two languages, giving in particular the explicit transformation of coordinates (further details are in [20]).

Let us consider the situation described in Fig. 3. We have considered two gauge choices, represented by the groups of diffeomorphisms φ and ψ, under which the point o in the physical manifold $\mathcal{M}_{\lambda,\Omega}$ corresponds to two different points in the background manifold $\mathcal{M}_0$, namely $q = \varphi^{-1}_{\lambda,\Omega}(o)$ and $p = \psi^{-1}_{\lambda,\Omega}(o)$. The transformation relating these two gauge choices is described by the two–parameter family of diffeomorphisms $\Phi_{\lambda,\Omega} = \varphi^{-1}_{\lambda,\Omega} \circ \psi_{\lambda,\Omega}$, so that $\Phi_{\lambda,\Omega}(p) = q$. This gauge transformation maps a tensor field T on $q \in \mathcal{M}_0$ to the tensor field $(\Phi^* T)(p) = \Phi^*(T(q))$ on $p \in \mathcal{M}_0$.

Now, let us consider a chart $(\mathcal{U}, \boldsymbol{X})$ on an open subset $\mathcal{U}$ of $\mathcal{M}_0$. The gauges $\varphi_{\lambda,\Omega}$ and $\psi_{\lambda,\Omega}$ define two maps from $\mathcal{M}_{\lambda,\Omega}$ to $\mathbb{R}^m$:

$$\begin{array}{rccc} \boldsymbol{X}\circ\varphi^{-1}_{\lambda,\Omega} : & \mathcal{M}_{\lambda,\Omega} & \longrightarrow & \mathbb{R}^m \\ & o & \longmapsto & x(q(o))\,, \end{array} \qquad \begin{array}{rccc} \boldsymbol{X}\circ\psi^{-1}_{\lambda,\Omega} : & \mathcal{M}_{\lambda,\Omega} & \longrightarrow & \mathbb{R}^m \\ & o & \longmapsto & x(p(o))\,. \end{array}$$

Then, we can look at the gauge transformation Φ in two different ways: from the *active* point of view or from the *passive* point of view. In the first case, one considers a diffeomorphism which changes the point on the background $\mathcal{M}_0$. To these points one associates different values of the coordinates in the chart $(\mathcal{U}, \boldsymbol{X})$. So the coordinate change is given by

$$x^\mu(p) \;\longrightarrow\; x^\mu(q) \tag{64}$$

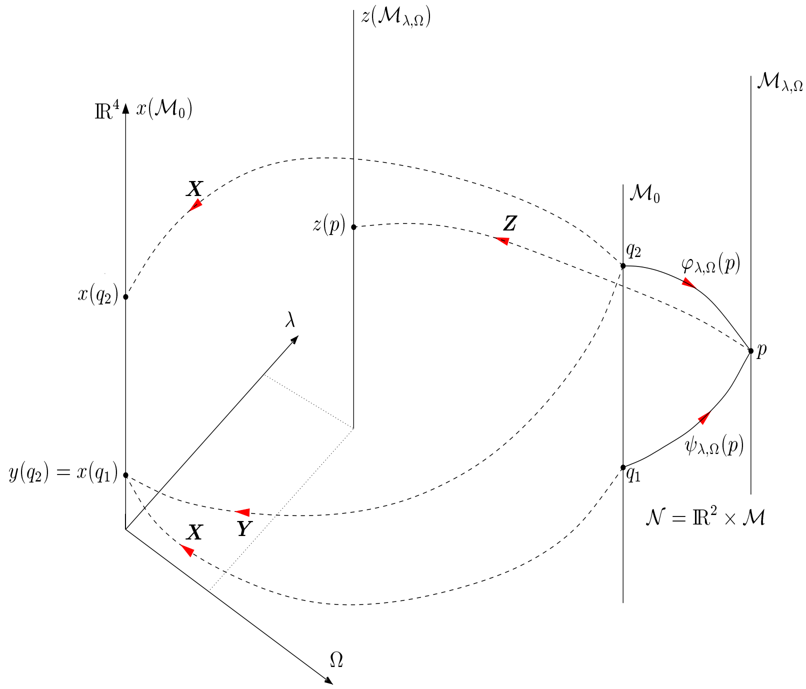

Figure 3. The picture shows how two different gauges φ and ψ, map the *physical* point $o \in \mathcal{M}_{\lambda\Omega}$ to two different points in the background manifold $\mathcal{M}_0$, namely q and p respectively. The coordinate version of this fact is shown on the left part of picture. We can look at this either from the *active* point of view (the two points q and p are given coordinates in the same chart by the map $\boldsymbol{X}$, namely $x^\mu(q)$ and $x^\mu(p)$), or from the *passive* point of view (the point q is given coordinates in two different charts by maps $\boldsymbol{X}$ and $\boldsymbol{Y}$, corresponding to two different points of $\mathrm{I\!R}^m$, $x(q)$ and $y(q) = x(p)$).

or, defining the pull–back of x as $\tilde{x}^\mu(p) := x^\mu(\Phi(p))$,

$$x^\mu(p) \longrightarrow \tilde{x}^\mu(p) \,. \tag{65}$$

If we instead consider the passive point of view, we need to introduce a new chart $(\mathcal{U}', y)$:

$$\boldsymbol{Y} := \boldsymbol{X} \circ \Phi^{-1}_{\lambda,\Omega} \,, \tag{66}$$

such that the two sets of coordinates are related by

$$y^\mu(q) = x^\mu(p) \,, \tag{67}$$

so we can say that the gauge transformation does not change the point on $\mathcal{M}_0$, but it changes the chart from $(\mathcal{U}, \boldsymbol{X})$ to $(\mathcal{U}', \boldsymbol{Y})$, i.e. the labels of the point of $\mathcal{M}_0$. The coordinate transformation is then

$$x^\mu(q) \longrightarrow y^\mu(q) \,. \tag{68}$$

Now, let us consider the transformation of a vector field V. From the active point of view, the components of V in the chart $(\mathcal{U}, \boldsymbol{X})$, V^μ, are related with the ones of the transformed vector field $\tilde{V}$, $\tilde{V}^\mu$, by

$$\tilde{V}^\mu = (\boldsymbol{X}_* \tilde{V})^\mu = (\boldsymbol{X}_* \Phi^*_{\lambda,\Omega} V)^\mu \,. \tag{69}$$

From the passive point of view, we can use the properties relating the pull–back and push–forward maps associated with diffeomorphisms:

$$\boldsymbol{X}_*\Phi^*_{\lambda,\Omega}V = \boldsymbol{X}_*\Phi^{-1}_{*\lambda,\Omega}V = \boldsymbol{Y}_*V\,, \tag{70}$$

so we get the well known result that the components of the transformed vector $\tilde{V}$ in the coordinate system $\boldsymbol{X}$ are defined in terms of the components of the vector V in the new coordinate system $\boldsymbol{Y}$:

$$\tilde{V}^\mu(x(p)) = (\boldsymbol{Y}_*V(q))^\mu = V'^\mu(y(q)) = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)\bigg|_{x(q)} V^\nu(x(q))\,. \tag{71}$$

In order to write down explicit expressions, we will apply the expansion of the pull–back of Φ^* [See equation (30)] to the coordinate functions x^μ. Then, the *active* coordinate transformation is given by

$$\begin{aligned}
\tilde{x}^\mu(p) &= x^\mu(q) = (\Phi^* x^\mu)(p)\\
&= x^\mu(p) + \lambda\xi^\mu_{(1,0)} + \Omega\xi^\mu_{(0,1)}\\
&\quad + \frac{\lambda^2}{2}\left(\xi^\mu_{(2,0)} + \xi^\nu_{(1,0)}\xi^\mu_{(1,0),\nu}\right) + \frac{\Omega^2}{2}\left(\xi^\mu_{(0,2)} + \xi^\nu_{(0,1)}\xi^\mu_{(0,1),\nu}\right)\\
&\quad + \lambda\Omega\left(\xi^\mu_{(1,1)} + \epsilon_0\xi^\nu_{(1,0)}\xi^\mu_{(0,1),\nu} + \epsilon_1\xi^\nu_{(0,1)}\xi^\mu_{(1,0),\nu}\right)\\
&\quad + \frac{\lambda^3}{6}\left(\xi^\mu_{(3,0)} + 3\xi^\nu_{(1,0)}\xi^\mu_{(2,0),\nu} + \xi^\rho_{(1,0)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,0),\nu}\right)\\
&\quad + \frac{\lambda^2\Omega}{2}\left(\xi^\mu_{(2,1)} + 2\xi^\nu_{(1,0)}\xi^\mu_{(1,1),\nu} + \xi^\nu_{(0,1)}\xi^\mu_{(2,0),\nu} + 2\epsilon_2\xi^\rho_{(1,0)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,0),\nu}\right.\\
&\quad \left. +(\epsilon_1-\epsilon_2)\xi^\rho_{(0,1)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,0),\nu} + (\epsilon_0-\epsilon_2)\xi^\rho_{(1,0)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,1),\nu}\right)\\
&\quad + \frac{\lambda\Omega^2}{2}\left(\xi^\mu_{(1,2)} + 2\xi^\nu_{(0,1)}\xi^\mu_{(1,1),\nu} + \xi^\nu_{(1,0)}\xi^\mu_{(0,2),\nu} + 2\epsilon_3\xi^\rho_{(0,1)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,1),\nu}\right.\\
&\quad \left. +(\epsilon_0-\epsilon_3)\xi^\rho_{(1,0)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,1),\nu} + (\epsilon_1-\epsilon_3)\xi^\rho_{(0,1)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,0),\nu}\right)\\
&\quad + \frac{\Omega^3}{6}\left(\xi^\mu_{(0,3)} + 3\xi^\nu_{(0,1)}\xi^\mu_{(0,2),\nu} + \xi^\rho_{(0,1)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,1),\nu}\right)\\
&\quad + \frac{\lambda^4}{24}\left(\xi^\mu_{(4,0)} + 4\xi^\nu_{(1,0)}\xi^\mu_{(3,0),\nu} + 3\xi^\nu_{(2,0)}\xi^\mu_{(2,0),\nu}\right.\\
&\quad \left. +6\xi^\rho_{(1,0)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(2,0),\nu} + \xi^\sigma_{(1,0)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,0),\nu})\right)\\
&\quad + \frac{\lambda^3\Omega}{6}\left(\xi^\mu_{(3,1)} + 3\xi^\nu_{(1,0)}\xi^\mu_{(2,1),\nu} + \xi^\nu_{(0,1)}\xi^\mu_{(3,0),\nu} + 3\epsilon_4\xi^\nu_{(2,0)}\xi^\mu_{(1,1),\nu}\right.\\
&\quad + 3\epsilon_5\xi^\nu_{(1,1)}\xi^\mu_{(2,0),\nu} + 3\xi^\rho_{(1,0)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,1),\nu} + 3(\epsilon_0\xi^\rho_{(1,0)}\xi^\nu_{(0,1),\rho} + \epsilon_1\xi^\rho_{(0,1)}\xi^\nu_{(1,0),\rho})\xi^\mu_{(2,0),\nu}\\
&\quad + (\epsilon_1-\epsilon_2-\epsilon_6)\xi^\sigma_{(0,1)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,0),\nu} + 3\epsilon_6\xi^\sigma_{(1,0)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,0),\nu}\\
&\quad \left. +3(\epsilon_2-\epsilon_6)\xi^\sigma_{(1,0)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,0),\nu} + (\epsilon_0-2\epsilon_2+\epsilon_6)\xi^\sigma_{(1,0)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,1),\nu}\right)\\
&\quad + \frac{\lambda^2\Omega^2}{4}\left(\xi^\mu_{(2,2)} + 2\xi^\nu_{(1,0)}\xi^\mu_{(1,2),\nu} + 2\xi^\nu_{(0,1)}\xi^\mu_{(2,1),\nu} + 2\xi^\nu_{(1,1)}\xi^\mu_{(1,1),\nu}\right.\\
&\quad + \epsilon_7\xi^\nu_{(2,0)}\xi^\mu_{(0,2),\nu} + \epsilon_8\xi^\nu_{(0,2)}\xi^\mu_{(2,0),\nu} + \xi^\rho_{(1,0)}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,2),\nu} + \xi^\rho_{(0,1)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(2,0),\nu}\\
&\quad + 4(\epsilon_0\xi^\rho_{(1,0)}\xi^\nu_{(0,1),\rho} + \epsilon_1\xi^\rho_{(0,1)}\xi^\nu_{(1,0),\rho})\xi^\mu_{(1,1),\nu}\\
&\quad - (\epsilon_3+\epsilon_2-\epsilon_1+\epsilon_9)\xi^\sigma_{(0,1)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(1,0),\nu}
\end{aligned}$$

$$
\begin{aligned}
& - (\epsilon_3 + \epsilon_2 - \epsilon_0 - \epsilon_9)\xi^\sigma_{(1,0)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,1),\nu} \\
& 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 - \epsilon_9)\xi^\sigma_{(1,0)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,1),\nu} \\
& 2(\epsilon_3 + \epsilon_2 - \epsilon_0\epsilon_1 + \epsilon_9)\xi^\sigma_{(0,1)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,0),\nu} \\
& -2(\epsilon_3 - \epsilon_0\epsilon_1)\xi^\sigma_{(1,0)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,0),\nu} - 2(\epsilon_2 - \epsilon_0\epsilon_1)\xi^\sigma_{(0,1)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,1),\nu}\Big) \\
& + \frac{\lambda\Omega^3}{6}\Big(\xi^\mu_{(1,3)} + 3\xi^\nu_{(0,1)}\xi^\mu_{(1,2),\nu} + \xi^\nu_{(1,0)}\xi^\mu_{(0,3),\nu} + 3\epsilon_{10}\xi^\nu_{(0,2)}\xi^\mu_{(1,1),\nu} \\
& + 3\epsilon_{11}\xi^\nu_{(1,1)}\xi^\mu_{(0,2),\nu} + 3\xi^\rho_{(0,1)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,1),\nu} + 3(\epsilon_0\xi^\rho_{(1,0)}\xi^\nu_{(0,1),\rho} + \epsilon_1\xi^\rho_{(0,1)}\xi^\nu_{(1,0),\rho})\xi^\mu_{(0,2),\nu} \\
& + (\epsilon_0 - \epsilon_3 - \epsilon_{12})\xi^\sigma_{(1,0)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,1),\nu} + 3\epsilon_{12}\xi^\sigma_{(0,1)}\xi^\rho_{(1,0),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,1),\nu} \\
& +3(\epsilon_3 - \epsilon_{12})\xi^\sigma_{(0,1)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(1,0),\rho}\xi^\mu_{(0,1),\nu} + (\epsilon_1 - 2\epsilon_3 + \epsilon_{12})\xi^\sigma_{(0,1)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(1,0),\nu}\Big) \\
& + \frac{\Omega^4}{24}\Big(\xi^\mu_{(0,4)} + 4\xi^\nu_{(0,1)}\xi^\mu_{(0,3),\nu} + 3\xi^\nu_{(0,2)}\xi^\mu_{(0,2),\nu} \\
& +6\xi^\rho_{(0,1)}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,2),\nu} + \xi^\sigma_{(0,1)}\xi^\rho_{(0,1),\sigma}\xi^\nu_{(0,1),\rho}\xi^\mu_{(0,1),\nu}\Big)\,, \qquad (72)
\end{aligned}
$$

where the vector fields $\xi^\mu_{(p,q)}$ and their derivatives are evaluated in $x(p)$. This expression gives the relation between the coordinates, in the chart $(\mathcal{U}, \boldsymbol{X})$, of the two points p and q of $\mathcal{M}_0$.

On the other hand, the *passive* coordinate transformation is found by inverting (72):

$$y^\mu(q) := x^\mu(p) = x^\mu(q) - \lambda\xi^\mu_{(1,0)}(x(p)) - \Omega\xi^\mu_{(0,1)}(x(p)) + O^2(\lambda, \Omega)\,, \qquad (73)$$

and then by expanding $x(p)$ around $x(q)$. We obtain in this way an expression of the form

$$y^\mu(q) = x^\mu(q) - \lambda\xi^\mu_{(1,0)}(x(q)) - \Omega\xi^\mu_{(0,1)}(x(q)) + O^2(\lambda, \Omega)\,, \qquad (74)$$

which gives the relation between the coordinates of any arbitrary point $q \in \mathcal{M}_0$ in the two charts $(\mathcal{U}, \boldsymbol{X})$ and $(\mathcal{U}', \boldsymbol{Y})$. Such a relation is needed to find the transformation of the components of a tensor field, by using (71), as it is usually done in textbooks for first order gauge transformations[26, 27]. However, in order to determine these transformation rules it is much simpler to apply directly the expressions (45-58), computing explicitly the Lie derivatives of the tensor field.

5. Conclusions

Many astrophysical systems (in particular, oscillating relativistic rotating stars) can be well described by perturbation theory depending on two parameters. A well–founded description of two–parameter perturbations can be very useful for such applications, specially in order to handle properly perturbations at second order and beyond.

In this paper we study the problem of gauge dependence of non–linear perturbations depending on two parameters, considering perturbations of arbitrary order in a geometrical perspective, and generalizing the results of the one–parameter case [2, 21] to the case of two parameters. We construct a geometrical framework in which a *gauge choice* is a two–parameter *group* of diffeomorphisms, while a *gauge transformation* is a two–parameter *family* of diffeomorphisms. We show that any two–parameter family of diffeomorphisms can be expanded in terms of Lie derivatives with respect to vectors $\xi^\mu_{(p,q)}$. In terms of this expansion, which can be derived order by order, we have general expressions for transformations of coordinates and tensor perturbations, and for the conditions for gauge invariance of tensor perturbations. We compute these expressions up to fourth order in the perturbative expansion, i.e. up to terms $\lambda^k \Omega^{k'}$ with $k + k' = 4$.

The way in which the expansion of a two-parameter family of diffeomorphisms was derived in this paper is order by order, constructing derivative operators that can be rewritten as Lie derivatives with respect some vector fields. The development of an underlying geometrical structure, analogous to the knight diffeomorphisms introduced in the one–parameter case [2], would be interesting for two reasons: first, in order to have a deeper mathematical understanding of the theory, and second, in order to derive a close formula, valid at all orders, for gauge transformations and gauge invariance conditions. The present paper has been devoted to the derivation of the useful formulae for practical applications. We leave the development of a more formal framework for future work.

Appendix A. Proof of the statement (29)

Let $\mathcal{L}$ be a derivative operator (defined for tensor fields of arbitrary rank on a differentiable manifold $\mathcal{M}$), in the sense that it is linear and satisfies the Leibniz rule. On functions, $\mathcal{L}$ defines a vector field ξ through the relation

$$\mathcal{L}\, f =: \xi(f), \qquad \forall f \in \mathcal{F}(\mathcal{M}) \; . \tag{A.1}$$

We want to prove that, on an arbitrary tensor field T,

$$\mathcal{L}\, T = \pounds_\xi\, T \; . \tag{A.2}$$

Clearly, (A.2) holds for a T of arbitrary rank iff it holds for a generic vector field X, which is the same of saying that

$$\mathcal{L}\, X = [\xi, X] \; . \tag{A.3}$$

Let us write (A.3) more explicitly as

$$(\mathcal{L}\, X)(f) = \xi(X(f)) - X(\xi(f)), \qquad \forall f \in \mathcal{F}(\mathcal{M}) \; . \tag{A.4}$$

Therefore, the proof of (A.2) is reduced to prove (A.4). To this end, let us consider the action of the operator $\mathcal{L}$ on the function $X(f)$. Of course, we have

$$\mathcal{L}(X(f)) = \xi(X(f)) \tag{A.5}$$

because of (A.1). Now, *if* we also assume that

$$\mathcal{L}(X(f)) = (\mathcal{L}\, X)(f) + X(\mathcal{L}\, f) \; , \tag{A.6}$$

putting (A.5) and (A.6) together and using (A.1) again, we get exactly (A.4). Thus, in order to establish (A.4) it is sufficient to prove (A.6), which is just a property of the operator $\mathcal{L}$ and does not involve ξ any longer. The left hand side of (A.6) can be rewritten as

$$\mathcal{L}(X(f)) = \mathcal{L}(\mathrm{d}f(X)) = \mathcal{L}(\mathcal{C}(\mathrm{d}f \otimes X)) \; , \tag{A.7}$$

where $\mathcal{C}$ represents a contraction. On the other hand we have, for the right hand side,

$$\begin{aligned}(\mathcal{L}\, X)(f) + X(\mathcal{L}\, f) &= \mathrm{d}f(\mathcal{L}\, X) + \mathrm{d}(\mathcal{L}\, f)(X) \\ &= \mathcal{C}(\mathrm{d}f \otimes \mathcal{L}\, X) + \mathcal{C}(\mathrm{d}(\mathcal{L}\, f) \otimes X) \\ &= \mathcal{C}\left(\mathrm{d}f \otimes (\mathcal{L}\, X) + \mathrm{d}(\mathcal{L}\, f) \otimes X\right) \; .\end{aligned} \tag{A.8}$$

From (A.7) and (A.8) we see that (A.6) is equivalent to the statement:

$$\mathcal{L}\left(\mathcal{C}(\mathrm{d}f\otimes X)\right)-\mathcal{C}\left(\mathcal{L}(\mathrm{d}f\otimes X)\right)+\mathcal{C}\left((\mathcal{L}\,\mathrm{d}f-\mathrm{d}(\mathcal{L}\,f))\otimes X\right)=0\,. \quad \text{(A.9)}$$

This holds provided that $\mathcal{L}$ satisfies two conditions:

(i) It must commute with a contraction, $\mathcal{L}\mathcal{C}=\mathcal{C}\mathcal{L}$;

(ii) On functions, it must commute with the differential operator d, so $\mathcal{L}\mathrm{d}f=\mathrm{d}(\mathcal{L}\,f)$.

Both of these are satisfied, since Φ^* does commute with $\mathcal{C}$ and d (see [28]).

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